Practice Midterm 1-solutions

No homework for the next week. Remember, the First exam will be on Wednesday, February 18 during the regular class time.

February 13, 2015

1 Problem 1. True or False.

Solution:

1. a. False.
2. b. False.
3. c. False.
4. d. True.
5. e. True.

2 Problem 2. Matrix multiplication.

Solution:

$$\begin{bmatrix} 0 & 1 & 5 & 1 \\ 1 & 2 & -3 & 0 \\ 1 & -1 & 3 & -1 \\ 9 & 7 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 34 \\ -11 \\ 8 \\ 41 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 1 & 3 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 10 \\ 5 & 10 \\ 2 & 6 \end{bmatrix}$$

3 Problem 3. Gauss-Jordan Approach.

Solution:

$$\begin{bmatrix}1&1&1&1&0&0\\1&2&2&0&1&0\\1&2&3&0&0&1\end{bmatrix} \to \begin{bmatrix}1&1&1&1&0&0\\0&1&1&-1&1&0\\0&1&2&-1&0&1\end{bmatrix} \to \begin{bmatrix}1&1&1&1&0&0\\0&1&1&-1&1&0\\0&0&1&0&-1&1\end{bmatrix} \to$$

$$\begin{bmatrix}1&1&1&1&0&0\\0&1&0&-1&2&-1\\0&0&1&0&-1&1\end{bmatrix} \to \begin{bmatrix}1&0&0&2&-1&0\\0&1&0&-1&2&-1\\0&0&1&0&-1&1\end{bmatrix} \implies A^{-1} = \begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}$$

4 Problem 4. Column Space and Null Space.

Solution: Since all columns are dependant, the column space is 1-dimensional:

$$C(A) = \lambda \begin{bmatrix}1\\3\\2\end{bmatrix} \quad \forall\lambda \in \mathbb{R}$$

In order to find the null space of A, we solve the equation $Ax = 0$. Since we have to solve $Ax = b$ we work with the augmented matrix $[A \quad b]$. Elimination gives

$$\begin{bmatrix}2&1&3&&1\\6&3&9&&3\\4&2&6&&2\end{bmatrix} \textit{elimination} \implies \begin{bmatrix}2&1&3&&1\\0&0&0&&0\\0&0&0&&0\end{bmatrix}.$$

The $rank(A) = rank([A \quad b]) = 1$. For $Ax = 0$ we have one equation $2x + y + 3z = 0$ and two free variables, y, z. Consequently we have two special solutions:

1. $y = 1, z = 0$ then $x = -\frac{1}{2}$
2. $y = 0, z = 1$ then $x = -\frac{3}{2}$.

The null space $N(A)$ is spanned by these special solutions:

$$N(A) = \alpha \begin{bmatrix}-\frac{1}{2}\\1\\0\end{bmatrix} + \beta \begin{bmatrix}-\frac{3}{2}\\0\\1\end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}$$

Next, we have $rank(A) = rank([A \quad b]) = 1$. Therefore $Ax = b$ is solvable. To solve the equation $Ax = b$ we need one particular solution.

For $Ax = b$ we have one equation $2x + y + 3z = 1$. A particular solution is $x = 0, y = 1, z = 0$.

To get a complete solution to $Ax = b$ we add this particular solution to the null space:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}.$$

5 Problem 5. Column Space and Null Space.

5.1 Define the column, row and the null spaces of the matrix A.

Solution:

Definition: Column Space is the vector space spanned by all columns of A. If we assume A has n columns which are denoted by $\boldsymbol{v_1}, \boldsymbol{v_2} \cdots, \boldsymbol{v_n}$, then

$$C(A) = \left\{ \sum_{i=1}^{n} c_i \boldsymbol{v_i} \middle| \forall c_i \in \mathbb{R}, i = 1, 2, \cdots, n \right\}$$

Definition: Row Space is the vector space spanned by all rows of A. If we assume A has m rows which are denoted by $\boldsymbol{u_1}, \boldsymbol{u_2} \cdots, \boldsymbol{u_m}$, then

$$Row(A) = C(A^T) = \left\{ \sum_{i=1}^{m} c_i \boldsymbol{u_i} \middle| \forall c_i \in \mathbb{R}, i = 1, 2, \cdots, m \right\}$$

Definition: Null Space is the vector space spanned by all solutions of $Ax = 0$, which is:

$$Null(A) = \{x | Ax = 0\}$$

5.2 Find basis of column, row and null space of A.

Solution:

We do (Gauss-Jordan) elimination to find pivot columns and pivot rows of A. These will give a basis of $C(A)$ and $C(A^T)$. We have:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The second and the fourth columns are the pivot columns. So the basis of $C(A)$ form the second and the fourth columns of A:

$$C(A) = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}.$$

A basis of $C(A^T)$ form the first and the second row of R:

$$C(A^T) = \alpha\ [0, 1, 2, 0, -2] + \beta[0, 0, 0, 1, 2], \quad \alpha, \beta \in \mathbb{R}.$$

In order to find the basis of the null space $N(A)$, we solve the equation $Ax = 0$. Using the (Gauss-Jordan) elimination above, we have

$$Ax = 0 \implies \begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The $rank(A) = 2$, the pivot variables are x_2, x_4 and we have

$(The\ number\ of\ columns) - rank(A) = dim(N(A)) = 5 - 2 = 3$. So, we have 3 free variables, x_1, x_3, x_5.

We have 2 equations:

$$\begin{cases} x_2 + 2x_3 - 2x_5 = 0 \\ x_4 + 2x_5 = 0 \end{cases} \implies \begin{cases} x_2 = -2x_3 + 2x_5 \\ x_4 = -2x_5 \end{cases}$$

We write three special solutions which form a basis of $N(A)$:

1. $x_1 = 1, x_3 = 0, x_5 = 0, \implies x_2 = 0, x_4 = 0$
2. $x_1 = 0, x_3 = 1, x_5 = 0, \implies x_2 = -2, x_4 = 0$
3. $x_1 = 0, x_3 = 0, x_5 = 1, \implies x_2 = 2, x_4 = -2$

A basis of $N(A)$ is:

$$N(A) = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$